Precalc Honor
Winkler

13-3 Arithmetic and Geometric Series

Name:__________

Arithmetic Series

An arithmetic series is formed by adding the terms of an arithmetic sequence:

1 + 2 + 3 + … + 38 + 39 + 40

So how do you add up all the terms without doing it one by one?

Write the terms out twice, once forwards and once backwards:

$$\begin{array}{cccccccccccc} 1 & + & 2 & + & 3 & + \dots + & 38 & + & 39 & + & 40 \\ \underline{40} & \underline{+} & \underline{39} & \underline{+} & \underline{38} & \underline{+ \dots +} & \underline{3} & \underline{+} & \underline{2} & \underline{+} & \underline{1} \\ 41 & & 41 & & 41 & & 41 & & 41 & & 41 \end{array}$$

Notice that the terms that line up all sum to 41.

That's 40 pairs that sum to 41, or 40(41). But notice that we've added every number in the series twice. The real sum is half as much:

$$S_{40} = \frac{40(41)}{2} = \frac{n(t_1 + t_{40})}{2}$$

Formula to find the sum, S_n, of the first n terms in an arithmetic sequence:

$S_n = \dfrac{n(t_1 + t_n)}{2}$. Substituting $t_n = t_1 + (n-1)\cdot d$ we get $S_n = \dfrac{n(t_1 + t_1 + (n-1)\cdot d)}{2}$ or $S_n = \dfrac{n(2t_1 + (n-1)\cdot d)}{2}$

Example: Find the sum of the first 12 positive even integers.

$\underbrace{2}_{t_1} + 4 + 6 + 8 + \cdots + \underbrace{24}_{t_{12}}$

$S_{12} = \dfrac{12(2+24)}{2} = 156$

Example: Given that the first five partial sums of an arithmetic sequence are $S_1 = 5$, $S_2 = 12$, $S_3 = 21$, $S_4 = 32$ and $S_5 = 45$, find the first five terms of the sequence and the explicit formula for the sequence.

$S_1 = t_1 = 5$ +7

$S_2 = t_1 + t_2 = 12$ +9

$S_3 = t_1 + t_2 + t_3 = 21$ +11

$S_4 = t_1 + t_2 + t_3 + t_4 = 32$ +13

$S_5 = \underbrace{t_1 + t_2 + t_3 + t_4}_{32} + t_5 = 45$

$\underset{t_1}{5}$ $\underset{t_2}{7}$ $\underset{t_3}{9}$ $\underset{t_4}{11}$ $\underset{t_5}{13}$

$t_n = 2n + 3$

Example: A theater has 60 seats in the first row, 68 seats in the second row, 76 seats in the third row, and so on in the same increasing pattern. If the theater has 20 rows of seats, how many seats are in the theater?

$$\underbrace{60}_{t_1} + \underbrace{68}_{t_2} + \underbrace{76}_{t_3} + \ldots + \underbrace{212}_{t_{20}}$$

$$S_{20} = \frac{20(60+212)}{2} = 2720$$

Example: A new business decides to rank its 9 employees by how well they work and pay them amounts that are in an arithmetic sequence with a constant difference of \$500 a year. If the total amount paid to the employees is to be \$250,000, what will the employees make per year?

$$t_n = t_1 + (n-1)d$$
$$= t_1 + (9-1)\cdot 500$$
$$= t_1 + 8(500)$$

$$S_n = \frac{n(t_1+t_n)}{2}$$

$$250{,}000 = \frac{9(t_1 + t_1 + 8(500))}{2}$$

$t_1 = \$25{,}777.78$ $t_2 = \$26{,}277.78 \ldots$

Geometric Series

A geometric series is formed by adding the terms of a geometric sequence:

$S_n = a + \cancel{ar^1} + \cancel{ar^2} + \ldots + \cancel{ar^{n-1}}$	Add the terms of the sequence.
$rS_n = r\left(a + ar^1 + ar^2 + \ldots + ar^{n-1}\right)$	Multiply both sides by r.
$rS_n = \cancel{ar^1} + \cancel{ar^2} + \ldots + \cancel{ar^{n-1}} + ar^n$	Distribute.
$S_n - rS_n = a - ar^n$	Subtract line 3 from line 1 canceling terms.
$S_n(1-r) = a\left(1-r^n\right)$	Pull the GCF out in front.
$S_n = \dfrac{a\left(1-r^n\right)}{(1-r)}$	Divide.

To find the sum, S_n, of the first n terms of a geometric sequence:

$$s_n = \frac{t_1(1-r^n)}{1-r}$$ Where t_1 the first term, $r \neq 1$ is the common ratio, and n is the number of terms.

Example: Find the sum of the first 8 terms of the sequence -5, 15, -45, 135, ...

$n = 8$; $t_1 = -5$, $r = -3$

$$S_8 = \frac{-5(1-(-3)^8)}{1-(-3)}$$

$$S_8 = \frac{-5(1-6561)}{4} = \frac{32800}{4} = 8200$$

Example: Find the sum of the first 15 terms of the following geometric sequence:

5, -10, 20, -40, 80, ...

$$S_{15} = \frac{t_1(1-r^n)}{1-r} = \frac{5(1-(-2)^{15})}{1 - -2}$$

$$= \frac{5(1+32768)}{3} = 54{,}615$$

Example: Given the series 6 - 14.4 + 34.56 - 82.944 + ..., approximate S_8 to the nearest tenth.

$$S_8 = \frac{t_1(1-r^n)}{1-r} = \frac{6(1-(-2.4)^8)}{1 - -2.4}$$

$$\frac{-14.4}{6} = \frac{34.56}{-14.4} = r = -2.4 \qquad S_8 = -1940.74$$

Example: To save for Ben's college education, his family invests \$2500 when he is born. Suppose that the investment earns 6% annual interest, compounded annually. How much will the investment be worth at the end of the 18th year?

Example: Find the sum of the first ten terms of the geometric series below:

$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = \_\_$$

$$S_{10} = \frac{4(1-(\frac{1}{2})^{10})}{1-\frac{1}{2}} = 7.99219$$